EM214 2.5

Solutions by substitutions

Homogenous DE's

A function $f(x,y)$ is homogenous of degree n if

$$f(ax, ay) = a^n f(x,y).$$

In practice, this means: for each term in f, the sum of the degrees of x and y must be the same.

Example • Are the following functions homogenous? If so, what is their degree?

- $f(x,y) = x^2y^3 + x^3y$

Not homogenous.

- $f(x,y) = x^2y^3 + 3x^4y$

Homogenous of degree 5

- $f(x,y) = \sqrt{xy^3} + x^2$

Homogenous of degree 2.

Solving a homogenous DE

To solve

$$A(x,y)\,dx + B(x,y)\,dy = 0$$

where A and B are both homogenous of the same degree, make the substitution

$$y = ux \qquad [\text{or } x = vy, \text{ your choice}]$$

You'll end up with a separable DE, which you can solve.

<u>Example</u> Solve: $\underbrace{(x^2+y^2)}_{A}dx + \underbrace{(x^2-xy)}_{B}dy = 0$

A, B homogenous of degree 2.

Try: $y(x,u) = ux$

$\therefore \quad dy = u\,dx + x\,du$

$\therefore \quad (x^2+u^2x^2)\,dx \quad + \quad (x^2-ux^2).(u\,dx + x\,du)$

$\therefore \quad (x^2+u^2x^2 + x^2u - u^2x^2)\,dx + (x^3 - ux^3)\,du = 0$

$\therefore \quad x^2(1+u)\,dx + x^3(1-u)\,du = 0$

$\therefore \quad \dfrac{x^2}{x^3}\,dx = -\dfrac{1-u}{1+u}\,du$

$\therefore \quad \int x^{-1}\,dx = -\int \dfrac{1-u}{1+u}\,du$

Let $w = 1+u \quad \therefore u = w-1$

$\therefore \quad dw = du$

$= \int \dfrac{1-(w-1)}{w}\,dw = \int \left(\dfrac{2}{w}-1\right)dw$

$= 2\ln|w| - w$

$= -\left[2\ln|1+u| - (u+1)\right]$

$$\therefore \quad \ln|x| = -\Big(\underbrace{2\ln|1+y/x|}_{\substack{=\ln\left|\frac{1}{x}\right||x+y| \\ = -\ln|x| + \ln|x+y|}} - y/x\Big) + C$$

$$\therefore \quad \ln|x| = 2\ln|x| - 2\ln|x+y| + y/x + C$$

$$\therefore \quad \ln\frac{|x+y|^2}{|x|} = \frac{y}{x} + C$$

$$\therefore \quad \frac{(x+y)^2}{|x|} = e^{y/x} e^{C} \qquad (C \in \mathbb{R})$$

$$\therefore \quad \frac{(x+y)^2}{|x|} = k e^{y/x} \qquad (k > 0)$$

Two cases

- $\underline{x > 0}$ $\quad \therefore \quad |x| = x$

$$\therefore \quad \frac{(x+y)^2}{x} = k e^{y/x} \qquad (k > 0)$$

or

- $\underline{x < 0}$ $\quad \therefore \quad |x| = -x$

$$\therefore \quad \frac{(x+y)^2}{x} = -k e^{y/x} \qquad (k < 0)$$

We can summarize both cases as:

$$\frac{(x+y)^2}{x} = A e^{y/x} \qquad (A \in \mathbb{R})$$

Example Solve:

$$(x^2+2y^2)\frac{dx}{dy} = xy \quad , \quad y(-1)=1$$

Soln Standard form:

$$\underbrace{(x^2+2y^2)}_{A}dx - \underbrace{xy}_{B}dy = 0$$

A,B homogenous of degree 2.

Try : $y = ux$

$\therefore dy = udx + xdu$

$$\therefore (x^2+2u^2x^2)dx - x^2u(udx+xdu) = 0$$

$$\therefore x^2(1+u^2)\,dx - x^3u\,du = 0$$

$$\therefore \frac{dx}{x} = \frac{u}{1+u^2}du$$

$$\therefore \int\frac{dx}{x} = \int\frac{u}{1+u^2}du$$

$$\therefore \ln|x| = \frac{1}{2}\ln|1+u^2| + C$$

$$\therefore \ln|x|^2 = \ln(1+u^2) + D$$

($1+u^2$ always >0)

$$\therefore x^2 = e^D(1+u^2)$$

($x^2>0$)

$$\therefore \quad x^2 = k(1+u^2) \qquad (k>0)$$

$$\therefore \quad x^2 = k\left(1 + y^2/x^2\right) \qquad (u = y/x)$$

$$\therefore \quad 1 + y^2/x^2 = Ax^2 \qquad (A = 1/k,\ A>0)$$

$$\therefore \quad \frac{y^2}{x^2} = Ax^2 - 1$$

$$\therefore \quad y^2 = x^2(Ax^2-1)$$

Sub in initial condition:

$$y(-1) = 1 \implies 1 = 1\cdot(A-1)$$

$$\therefore \quad A = 2$$

$$\therefore \quad y^2 = x^2(2x^2-1)$$

Can simplify further:

$$y = \pm|x|\sqrt{2x^2-1}$$

Need : $y(-1) = 1$

So $x<0$

$\therefore |x| = -x$

and we need $+\sqrt{}$

$$\therefore \quad y = -x\sqrt{2x^2-1}$$

<u>Example</u> Solve $\dfrac{dy}{dx} = \dfrac{x+3y}{3x+y}$, $y(1) = 2$

$$(3x+y)\,dy = (x+3y)\,dx \qquad \text{Homogenous DE of degree 1.}$$

Write $x(u,y) = uy$

$$\therefore \quad dx = y\,du + u\,dy$$

$$\therefore \quad (3uy+y)\,dy = (uy+3y)(y\,du + u\,dy)$$

$$\therefore \quad \left[3uy + y - u(uy+3y)\right] dy = (uy+3y)\,y\,du$$

$$\therefore \quad \left[y - u^2 y\right] dy = y^2(u+3)\,du$$

$$\therefore \quad y(1-u^2)\,dy = y^2(u+3)\,du$$

$$\therefore \quad \int y^{-1}\,dy = \underbrace{\int \frac{u+3}{1-u^2}\,du}$$

$$\frac{u+3}{1-u^2} = \frac{A \cdot 1}{1+u} + \frac{B \ 2}{1-u}$$

$$\therefore \quad A(1-u) + B(1+u) = u+3$$

$$\therefore \quad A + B + u(-A+B) = u+3$$

$$\therefore \quad A+B = 3, \quad -A+B = 1$$

$\therefore \quad 2B=4 \quad B=2$

$A=1$

$$= \frac{1}{1+u} + \frac{2}{1-u} = \frac{1-u+2(1+u)}{1-u^2} = \frac{3+u}{1-u^2}$$

$$\therefore \int \left(\frac{1}{1+u} + \frac{2}{1-u}\right) du = \ln|1+u| - 2\ln|1-u|$$

$$\therefore \ln|y| = \ln|1+u| - 2\ln|1-u| + C$$

Our I.C. says $y(1)=2$ i.e. both x and y are positive $\therefore |y|=y$, $|1+x/y| = 1+x/y$

$$\therefore \ln|y| = \ln|1+x/y| - 2\ln|1-x/y|$$

$$\therefore y = (1+x/y)(1-x/y)^{-2} C$$

$$= \frac{1}{y}(y+x)\, y^2 (y-x)^{-2} \cdot C$$

$$\therefore 1 = C(y+x)(y-x)^{-2}$$

$$y(1) = 2 \Rightarrow 1 = C(3)(1)^{-2}$$

$$\therefore 3C = 1 \therefore C = 1/3$$

i.e.

$$\therefore 1 = 1/3\,(y+x)(y-x)^{-2}$$

$$(x^2+y^2)\,dx + (x^2-xy)\,dy = 0$$

Try to solve with I.F. ?

$I = I(x)$?

$$\underbrace{I(x)(x^2+y^2)}_{A}dx + \underbrace{I(x)(x^2-xy)}_{B}dy = 0$$

$$A_y = B_x \iff I\cdot 2y = I'(x^2-xy) + I\cdot(2x-y)$$

$$\therefore\ I'(x) = I\cdot\left(\frac{2y-(2x-y)}{x^2-xy}\right)$$

$$= I\cdot\left(\frac{3y-2x}{x^2-xy}\right) \times$$

$I = I(y)$?

$$A_y = B_x \iff I'(y)(x^2+y^2) + I(y)(2y) = I(y)(2x-y)$$

$$\therefore\ I'(y) = I(y)\left(\frac{2x-y-2y}{x^2+y^2}\right) \times$$

Solving a Bernoulli DE

$$\frac{dy}{dx} + P(x)y = f(x)y^n \quad \ldots \text{ Bernoulli DE}$$

Make substitution : $u = y^{1-n}$. Then it becomes a linear DE, when you can use integrating factor.

Example Solve

$$\frac{dy}{dx} - y = e^x y^2$$

Solution Bernoulli DE with $P=-1$, $f(x) = e^x$, $n=2$.

Let $u = y^{1-2} = y^{-1}$. i.e. $y = u^{-1}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -u^{-2} \frac{du}{dx}$$

$$\therefore -u^{-2}\frac{du}{dx} - u^{-1} = e^x u^{-2}$$

$$\therefore \frac{du}{dx} + u = -e^x \quad \ldots \text{ linear DE}$$

$$I = e^{\int P\,dx} = e^{\int dx} = e^{x}$$

$$\therefore \quad e^{x}\left[u' + u\right] = -e^{x} \cdot e^{x}$$

$$\therefore \quad \frac{d}{dx}\left[e^{x}u\right] = -e^{2x}$$

$$\therefore \quad e^{x}u = -\int e^{2x}\,dx$$

$$= -\tfrac{1}{2}e^{2x} + C$$

$y = u^{-1}$

$$\therefore \quad u = -\tfrac{1}{2}e^{x} + Ce^{-x}$$

$$\therefore \quad y = \frac{1}{-\tfrac{1}{2}e^{x} + Ce^{-x}}$$

Finally, if the ODE has the form

$$\frac{dy}{dx} = f(Ax + By + C) \qquad A, B, C, D \text{ constants}$$

eg. $$\frac{dy}{dx} = (2x + 3y + 7)^4 + 5(2x + 3y + 7)^2 + 12$$

or $$\frac{dy}{dx} = \cos(2x + 3y + 7) - 14$$

then make the substitution

$$u = Ax + By + C$$

to reduce it to a separable DE.

<u>Example</u> Solve $$\frac{dy}{dx} = (-2x + y)^2 - 7$$

Let $u(x) = -2x + y$

$$\therefore \frac{du}{dx} = -2 + \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + 2$$

$$\therefore \frac{du}{dx} + 2 = u^2 - 7$$

$$\therefore \frac{du}{dx} = u^2 - 9$$

$$\therefore \quad \frac{du}{(u+3)(u-3)} = dx$$

$$\frac{1}{(u+3)(u-3)} = \frac{A}{u+3} + \frac{B}{u-3}$$

$$A(u-3) + B(u+3) = 0.u + 1$$

$$\therefore \quad (A+B)u + (-3A+3B) = 0.u + 1$$

$$\therefore \quad A+B=0, \qquad -3A+3B=1$$

$$\therefore \quad -6A = 1 \implies A = -1/6$$

$$B = 1/6$$

$$\therefore \quad 1/6 \int \left[\frac{-1}{u+3} + \frac{1}{u-3} \right] du = x + C$$

$$\therefore \quad -\ln(u+3) + \ln(u-3) = 6x + 6C$$

$$\therefore \quad \frac{u-3}{u+3} = ke^{6x} \qquad (\underset{>0}{k} = e^{6C})$$

Solve for y: $\quad (u-3) = ke^{6x}(u+3)$

$$\therefore \; u\left(1 - ke^{6x}\right) = 3\left(ke^{6x} + 1\right)$$

$$u = \frac{3\left(ke^{6x} + 1\right)}{1 - ke^{6x}}$$

$\therefore \; y - 2x = $ " "

$$\therefore \; y = 2x + \frac{3\left(ke^{6x} + 1\right)}{1 - ke^{6x}}$$